TIME-DOMAIN ASTRONOMY

Lectures 6: Wavelet Analysis

Stefano Covino
INAF / Brera Astronomical Observatory

PSD Varying Time-Series

- The trigonometric basis functions used in the Fourier transform have an infinite extent and for this reason the Fourier transform may not be the best method to analyze non periodic time series data, such as the case of a localized event (e.g., a burst that decays over some time scale so that the PSD is also varying with time).

- We can (and actually do) evaluate the PSD for finite stretches of time series in order to detect its changes.

- This approach (called spectrogram, or dynamical power spectra analysis, or even windowed Fourier transform) suffers from degraded spectral resolution and is sensitive to the specific choice of time series segmentation length.

Windowed Fourier Transform

- The FT is performed on a sliding segment of length T from a time series of time step δt and total length $N\delta t$, thus returning frequencies from T^{-1} to $(2\delta t)^{-1}$ at each time step.
- The segments can be windowed with an arbitrary function such as a boxcar (no smoothing) or any other choice.
- The WFT represents an inaccurate and inefficient method of time–frequency localization, as it imposes a scale or "response interval" T into the analysis.
- The inaccuracy arises from the aliasing of high- and low-frequency components that do not fall within the frequency range of the window. The inefficiency comes from the $T/(2\delta t)$ frequencies, which must be analyzed at each time step.

Wavelets

- A popular family of basis functions is called wavelets.
  - By construction, wavelets are localized in *both* frequency and time domains.
- Individual wavelets are specified by a set of *wavelet filter coefficients*. Given a wavelet, a complete orthonormal set of basis functions can be constructed by scalings and translations.
- Different wavelet families trade the localization of a wavelet with its smoothness. Popular wavelets include "Mexican hat", Haar and Daubechies wavelets.

Wavelets

- A wavelet is a "small wave," where small derives from the fact that it is mostly limited to an interval of time.
- A (mother) wavelet ψ has to satisfy two requirements: its integral must be zero and the integral of its square must be unity:

$$\int_{-\infty}^{\infty} \psi(t)dt = 0 \qquad \int_{-\infty}^{\infty} \psi^2(t)dt = 1$$

- It is easy to see that these requirements imply that $\psi(t)$ is essentially non-zero only over a limited range of t and that it has to extend both above and below zero.

Wavelets

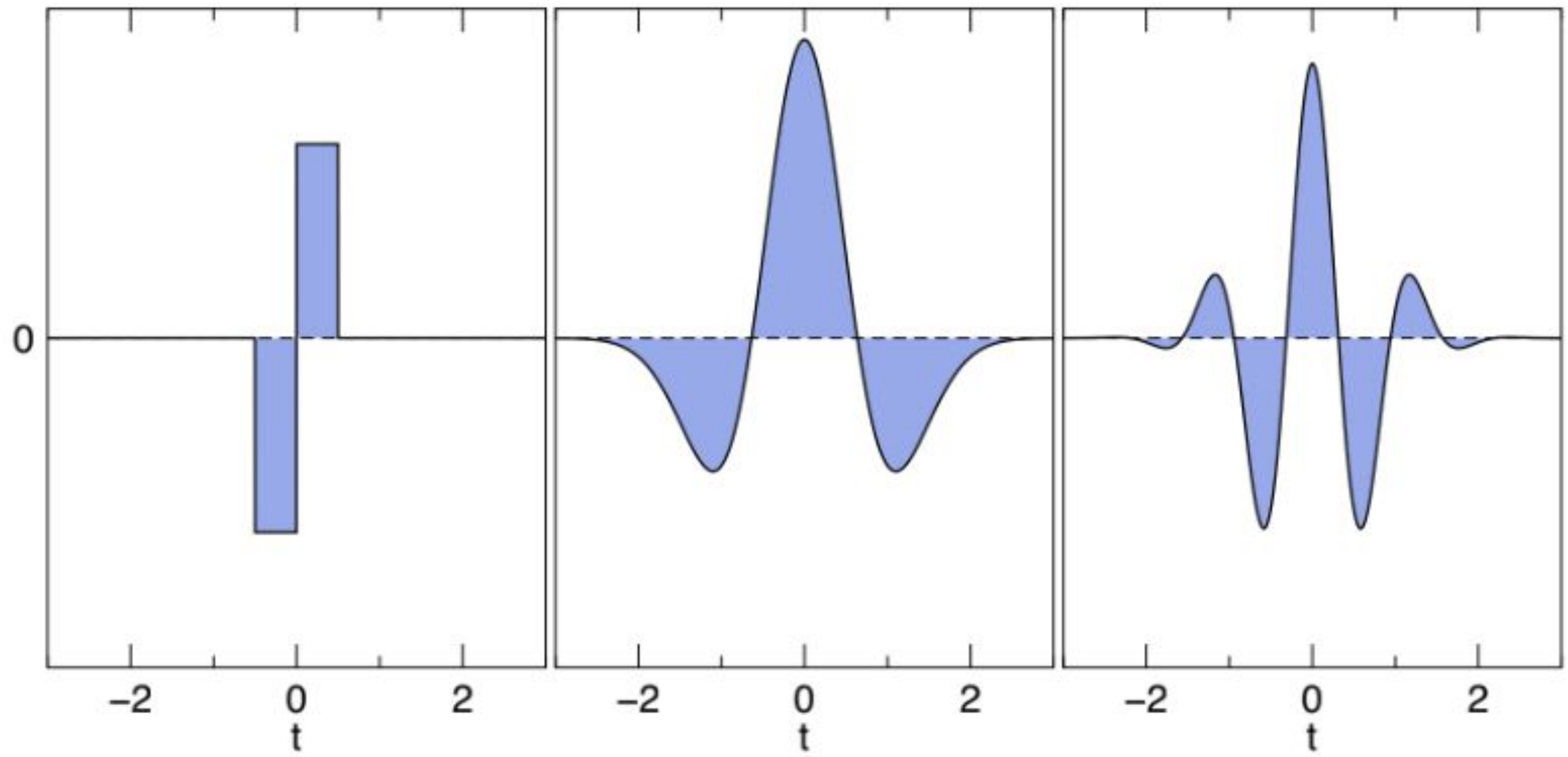


Fig. 16: Three examples of wavelets. Left panel: Haar Wavelet. Middle panel: mexican hat wavelet. Right panel: Morlet wavelet.

Wavelets

- A wavelet can be shifted by τ and dilated by a scale parameter σ:

$$\psi_{\tau,\sigma}(t) = \frac{1}{\sqrt{\sigma}}\psi\left(\frac{t-\tau}{\sigma}\right)$$

- Again $\psi(t)$ is essentially non-zero only over a limited range of t and that it has to extend both above and below zero.

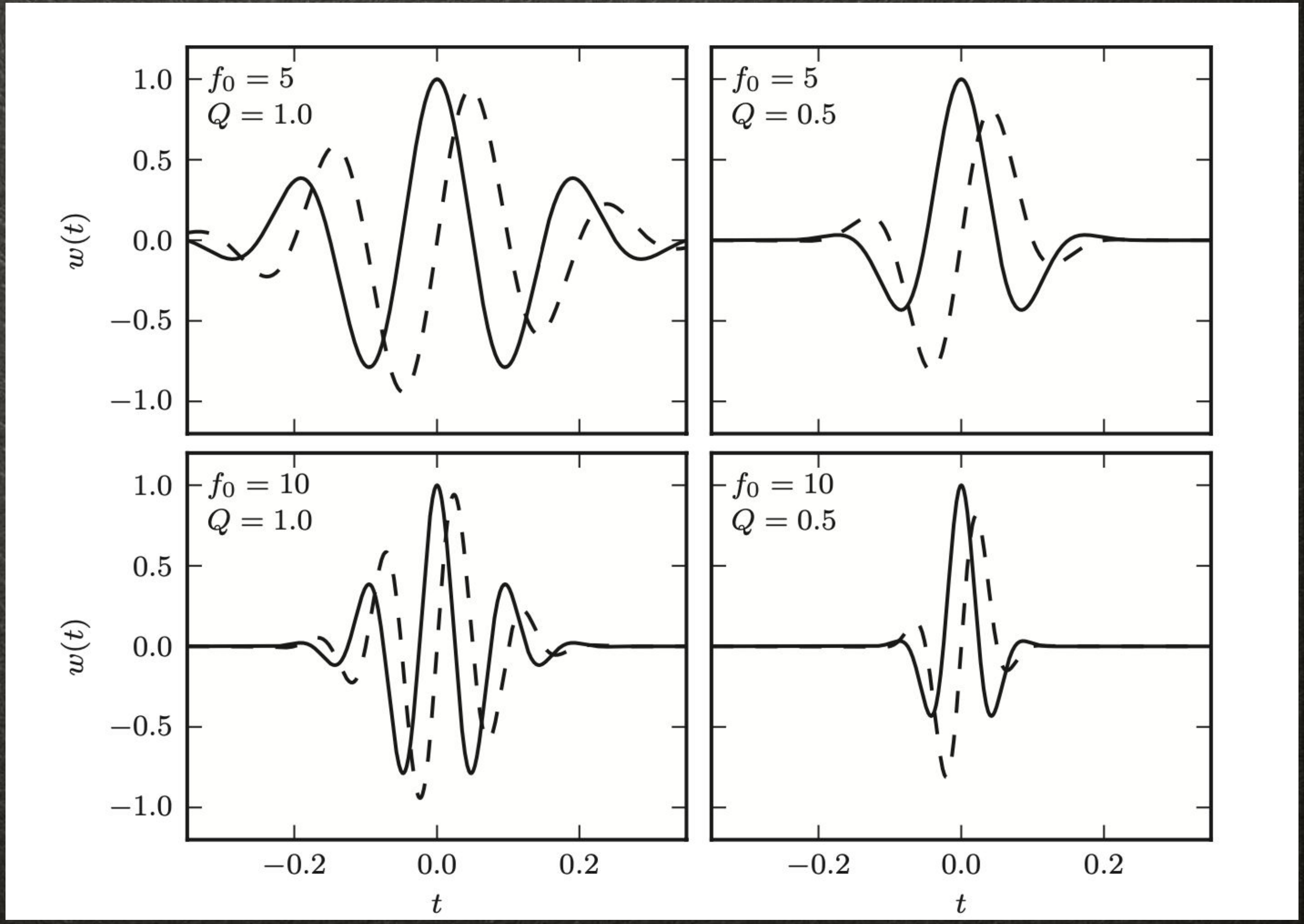
- The wavelet transform of a function f(t) is computed by correlating f(t) with the complex conjugate of $\psi_{\tau,\sigma}(t)$ (wavelets can also be complex functions):

$$W[f(\tau,\sigma)] = \int_{-\infty}^{\infty} f(t)\psi^{*}_{\tau,\sigma}(t)dt$$

Wavelets

- The discrete wavelet transform (DWT) can be used to analyze the power spectrum of a time series as a function of time.

- A possible wavelet might be: $$w(t|t_0,\, f_0,\, Q) = A\, \exp[i 2\pi f_0 (t - t_0)]\, \exp[-f_0^2 (t - t_0)^2 / Q^2],$$



- where t_0 is the central time, f_0 is the central frequency, and the dimensionless parameter Q is a model parameter which controls the width of the frequency window.

Wavelets

- The Fourier transform of the previous wavelet is:

$$W(f|t_0, f_0, Q) = \left(\frac{\pi}{f_0^2/Q^2}\right)^{1/2} \exp(-i2\pi f t_0) \exp\left[\frac{-\pi^2 Q^2 (f - f_0)^2}{Q f_0^2}\right].$$

- Technically speaking, this is not exactly a wavelet. It is indeed closely related to a true wavelet, the *Morlet wavelet*, through a simple scaling and offset.

- One might think to it as a "matched filters".

Wavelets

- Given a signal $h(t)$, its wavelet transform is given by the convolution:

$$H_w(t_0; f_0, Q) = \int_{-\infty}^{\infty} h(t)\, w(t|t_0,\, f_0,\, Q).$$

- By the convolution theorem, we can write the Fourier transform of H_W as the pointwise product of the Fourier transforms of $h(t)$ (e.g. by a DFT) and of $w(t; t_O, f_O, Q)$.

- Then, the wavelet PSD, can be defined by:

$$\mathrm{PSD}_{\mathrm{w}}(\, f_O, t_O; Q) = |H_w(t_O; f_O, Q)|^2$$

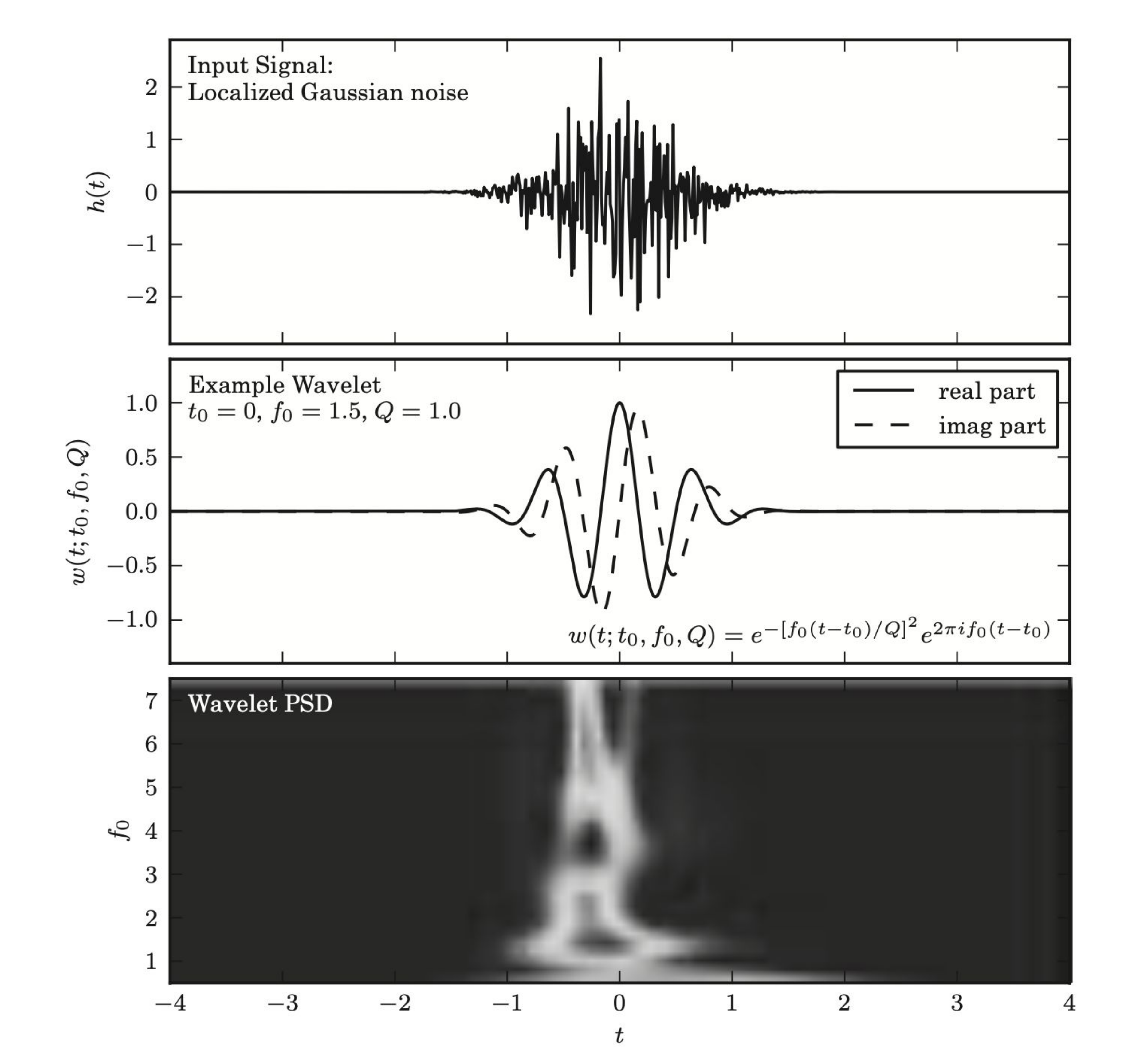
Wavelets

- As said, unlike the typical Fourier-transform PSD, the wavelet PSD allows detection of frequency information which is localized in time.

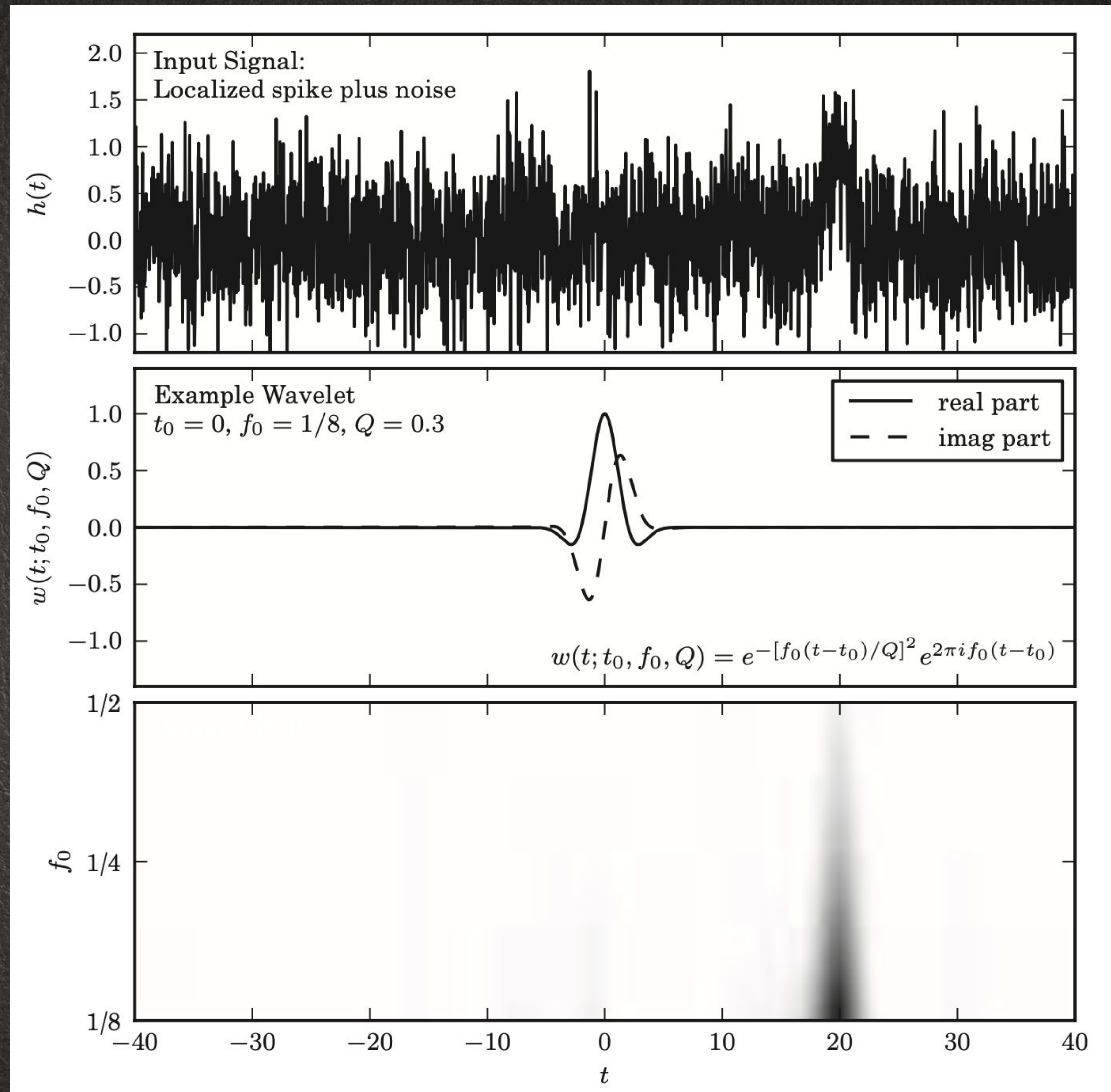
- This is indeed one the approaches (we are simplifying a lot here…) used in the, e.g., LIGO/Virgo projects to detect gravitational wave events.

  - Because of the noise level in the GW measurements, rather than a standard wavelet they instead use functions which are tuned to the expected form of the signal (i.e., matched filters).

- Example with a localized Gaussian noise.
- The middle panel shows an example wavelet.
- The lower panel shows the power spectral density as a function of the frequency f_0 and the time t_0, for Q = 1.0.



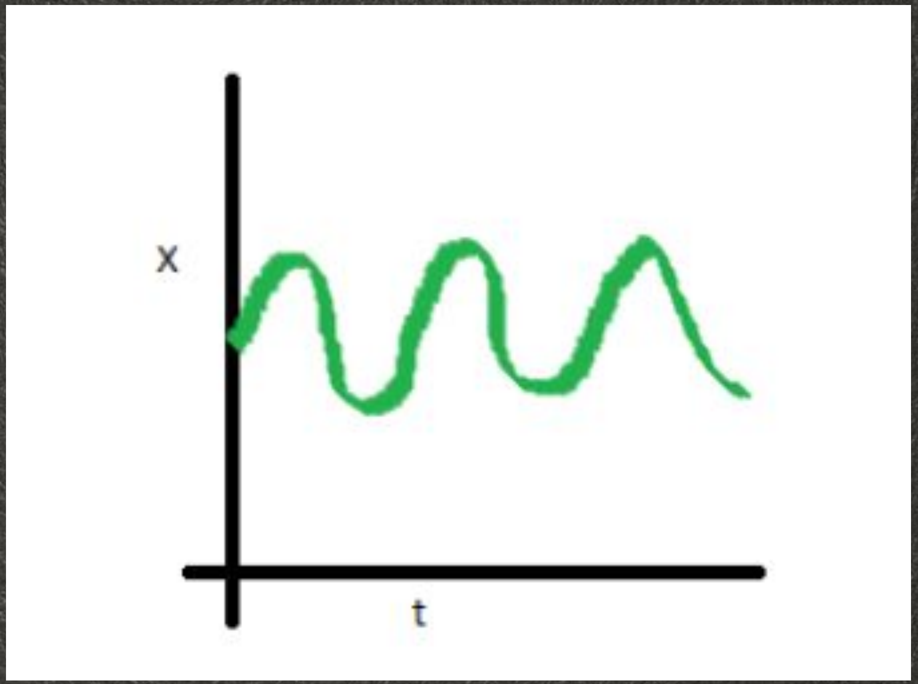
- The upper panel shows the input signal, which consists of a Gaussian spike in the presence of white (Gaussian) noise.
- The middle panel shows an example wavelet.
- The lower panel shows the PSD as a function of the frequency f_0 and the time t_0, for $Q = 0.3$.



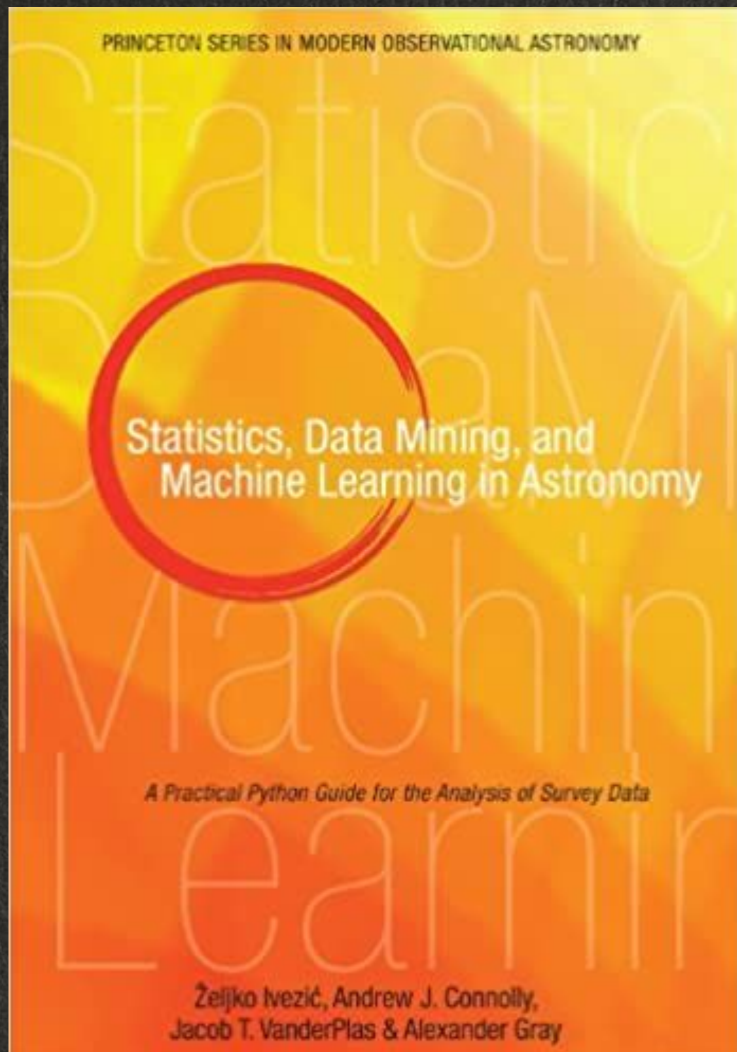
Exercise

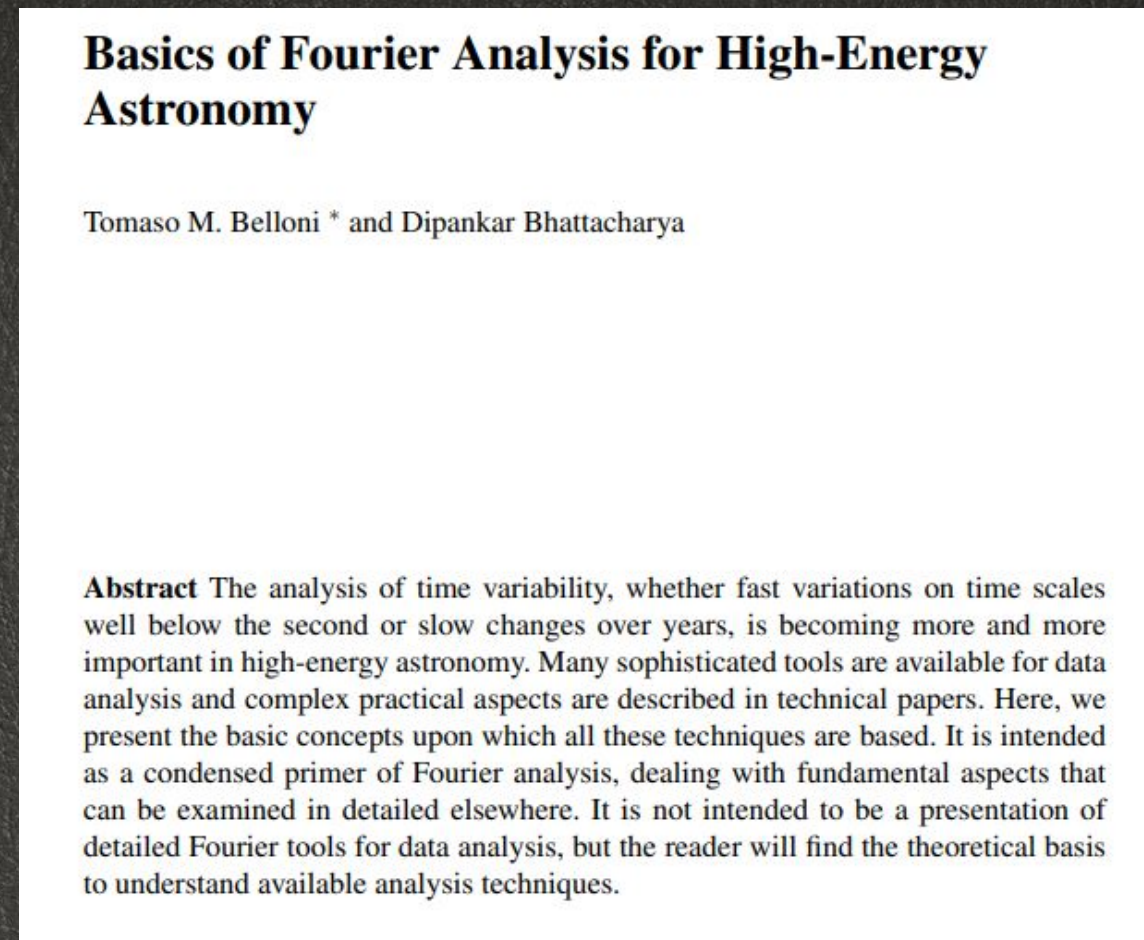
- Useful notebook:

1. Wavelets



REFERENCES AND DEEPENING





Basics of Fourier Analysis for High-Energy Astronomy

Tomaso M. Belloni * and Dipankar Bhattacharya

Abstract The analysis of time variability, whether fast variations on time scales well below the second or slow changes over years, is becoming more and more important in high-energy astronomy. Many sophisticated tools are available for data analysis and complex practical aspects are described in technical papers. Here, we present the basic concepts upon which all these techniques are based. It is intended as a condensed primer of Fourier analysis, dealing with fundamental aspects that can be examined in detailed elsewhere. It is not intended to be a presentation of detailed Fourier tools for data analysis, but the reader will find the theoretical basis to understand available analysis techniques.

Tomaso Belloni

Zeljko Izevic et al.

Dipankar Bhattacharya